2DFTs as a Functor

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November 22, 2023

Outline

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Categories

High-level Idea

Category theory is an abstract theory of functions.

Definition (Category)

A *category* $\mathcal{C}$ consists of a collection of *objects* $A, B, C, \ldots$ and a collection of *arrows* $f, g, h, \ldots$ such that

- For any arrow f, there are objects $\text{dom}(f)$ and $\text{cod}(f)$, called the *domain* and *codomain* of f respectively. We write $f : A \to B$ when $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
- Given any two arrows $f : B \to C$, $g : A \to B$, there is an arrow $f \circ g$, called the *composition* of f and g. Composition is associative.
- Given any object A, there is an *identity arrow* on A $\text{id}_A : A \to A$, which are identities w.r.t. composition.

Categories (Examples)

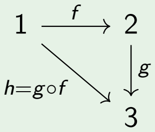
Example

Most mathematical objects can be bundled into categories: **Set**, **Grp**, $\mathbf{Vect}_{\mathbb{R}}$, ...

Example

Various objects can be thought of as categories: monoids, posets, ...

Example (3)



Functors

Definition (Functor)

Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of mappings F_1 on objects and F_2 on arrows such that

- Given any $\mathcal{C}$-object X, $F_1[X]$ is a $\mathcal{D}$-object.
- Given any $\mathcal{C}$-arrow $f : A \to B$, there is a $\mathcal{D}$-arrow $F_2[f] : F_1[A] \to F_1[B]$, such that F_2 satisfying:
  - For all $\mathcal{C}$-objects X,
    $$F_2[\mathrm{id}_X] = \mathrm{id}_{F_1[X]}\,.$$
  - For all $\mathcal{C}$-arrows $f : B \to C$ and $g : A \to B$,
    $$F_2[f] \circ F_2[g] = F_2[f \circ g].$$

Usually we use F for both F_1 and F_2.

Functorial / Categorical Semantics

High-level Idea

We will often view functors from $\mathcal{C}$ to $\mathcal{D}$ as *models* or *interpretations* of the *theory* $\mathcal{C}$ in $\mathcal{D}$.

Example

A monoid $\mathcal{M}$ is a category with one object. A functor $\mathcal{M} \to \mathbf{Set}$ is a monoid action.

The "Theory" of Automata

Definition (Shape$_\Sigma$)

For any finite alphabet Σ, there is a three object category $\textbf{Shape}_A$ generated by the following finite graph, where there is one morphism for each $a \in A$.

$$\text{in} \xrightarrow{\ \triangleright\ } \overset{\overset{a}{\circlearrowright}}{\text{states}} \xrightarrow{\ \triangleleft\ } \text{out}$$

We represent words $w \in \Sigma^\star$ by a composition of arrows, e.g., 01100 by $\triangleright; 0; 1; 1; 0; 0; \triangleleft$, where $f ; g = g \circ f$.

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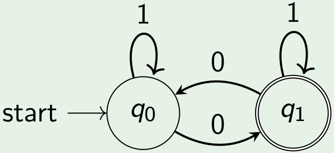
DFAs

Definition (Deterministic Finite Automata)

A *(deterministic) finite automaton* $\mathcal{T}$ is a tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set of *states*,
- Σ is a finite alphabet,
- $\delta : Q \times \Sigma \to Q$ is the *transition function*
- $q_0 \in Q$ is the *initial state*,
- $F \subseteq Q$ is the set of *final states*.

Example



DFAs as a Functor

Definition (DFA (revised))

A *(deterministic) finite automaton* with input alphabet Σ is a functor $F : \mathbf{Shape}_\Sigma \to \mathbf{Set}$ such that $F(\mathrm{in}) = \{\bullet\}$, $F(\mathrm{states})$ is non-empty and $F(\mathrm{out}) = \{\mathrm{false}, \mathrm{true}\}$.

We recover the previous definition by setting

- $Q = F(\mathrm{states})$,
- $q_0 = F(\triangleright)(\bullet)$
- $\delta(a, -) = F(a)$
- $x \in F$ iff $F(\triangleleft)(x) = \mathrm{true}$.

Given a word $w = \triangleright; w_1; \ldots; w_n; \triangleleft$ then $F(w) : \{\bullet\} \to \{\mathrm{false}, \mathrm{true}\}$ is the constantly true function if and only if w is in the language recognised by the DFA.

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2DFTs

Definition (Two-way Deterministic Transducer)

A *two-way deterministic transducer (2DFT)* $\mathcal{T}$ is a tuple $(Q, \rho, \Sigma, \Gamma, \delta, q_0, F)$, where

- Q is a finite set of *states*,
- $\rho : Q \to \{-1, 1\}$ is a *direction map*[a],
- Σ is a finite *input alphabet*,
- Γ is a finite *output alphabet*,
- $\delta : \Sigma \sqcup \{\triangleright, \triangleleft\} \to (Q \rightharpoonup \Gamma^\star \times Q)$ is the *transition function*
- $q_0 \in Q^\rightarrow$ is the *initial state*,
- $F \subseteq Q$ is a set of *final states*.

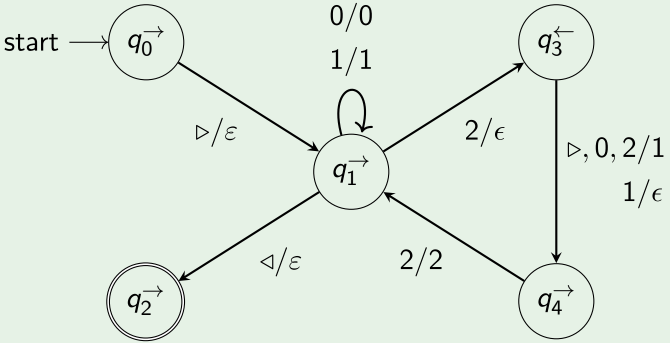
[a] We write $q^\rightarrow$ if $\rho(q) = 1$ and $q^\leftarrow$ if $\rho(q) = 1$

A 2DFT $\mathcal{T}$ defines a partial function $[\![\mathcal{T}]\!] : \Sigma^\star \to \Gamma^\star$ where the input string $\triangleright w_1 \ldots w_n \triangleleft$ is sent to a "valid" sequence of configurations by δ.

2DFT Example

Example

The following 2DFT takes any string and ensures that every 2 is preceded by a 1 by adding 1s if necessary.



2RFTs

For now, we will focus on a specific subclass of 2DFTs.

Definition (Two-way Reversible Transducer)

A *two-way reversible transducer (2RFT)* $\mathcal{T}$ with input alphabet Σ and output alphabet Γ is a 2DFT such that

- F is a singleton
- $\delta(a)$ is a partial *injection* for each $a \in \Sigma$.

Transition Diagrams

Compared with DFAs we have more structure here

- We can go forwards *and backwards* along the tape
- We need some way to “output” strings
- We require injectivity

This is solved by introducing a new category of “transition diagrams” **TransDiag**.

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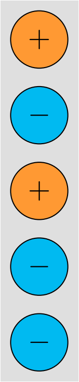
Objects

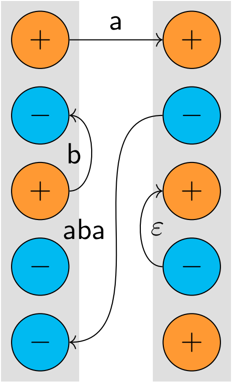
Objects are binary[1] words[2], which we write vertically top to bottom



[1]We write the objects as + and − rather than 1 and 0
[2]technically, multisets

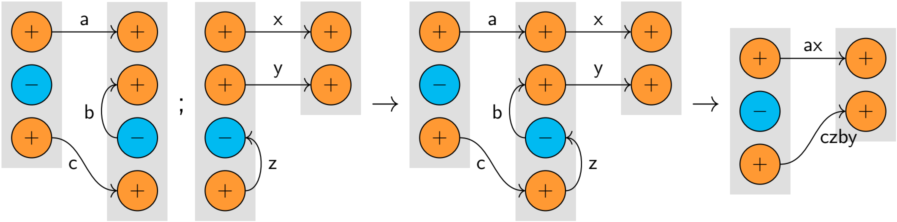
Morphisms

Morphisms are special "diagrams" between these words subject to some restrictions regarding polarity and vertex degree.
Edges are labelled with strings in the output alphabet.



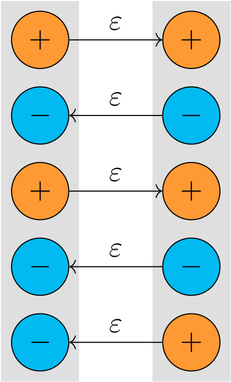
Composition

In a nutshell, glue them together and then concatenate strings along the path[3].



[3]If a path went into the “middle” it just disappears in the output

Identities



Category Extras

You do not have to understand this slide!

This is not just a category, it is a compact-closed category:

- Monoidal structure comes from taking disjoint union of objects / diagrams,
- Objects have duals by flipping polarity and reversing,
- "Cups" and "caps" come from "bending" the identity morphisms.

We can think of the category as being freely generated by arrows of the form $+ \xrightarrow{a} +$.

Strikingly similar to Temperley-Lieb Categories from physics / knot theory.

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2RFTs as a Functor

Definition (Two-way Reversible Transducer)

A *two-way reversible transducer (2RFT)* $\mathcal{T}$ with input alphabet Σ and output alphabet Γ is a functor $F : \mathbf{Shape}_\Sigma \to \mathbf{TransDiag}$ such that

- $F(\mathrm{states})$ is a non-empty binary word corresponding to the *ordered* directed set of states of the transducer.
- $F(\mathrm{in})$ and $F(\mathrm{out})$ are both the binary word $+$.

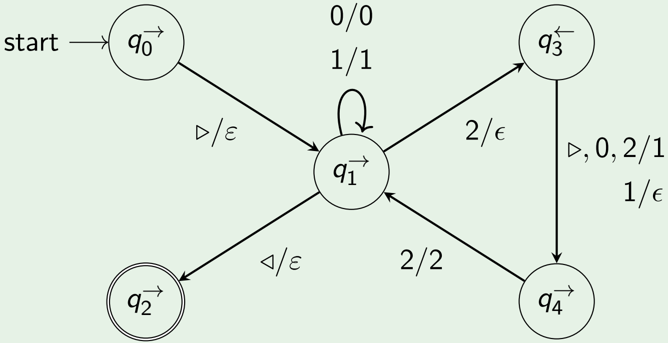
For any word $w \in \Sigma^\star$ we have a planar diagram $F(w) : + \to +$ which is either the empty diagram or a single path with label l. This corresponds to the partial function $[\![\mathcal{T}]\!] : \Sigma^\star \rightharpoonup \Gamma^\star$.

2RFT-Functor Example

Let's take the same example from before and find the equivalent functor.

Example

The following 2RFT takes any string and ensures that every 2 is preceded by a 1 by adding 1s if necessary.



2RFT-Functor Example (cont.)

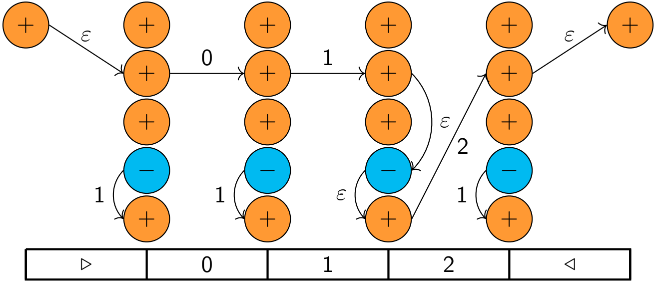
It has 5 states: $q0^{\rightarrow}$, $q1^{\rightarrow}$, $q2^{\rightarrow}$, $q3^{\leftarrow}$, $q4^{\rightarrow}$. Using that order and assigning $+$ to the forward vertices, i.e., $q_i^{\rightarrow}$, and $-$ to the backward vertices, i.e., $q_j^{\leftarrow}$, we obtain the word $F(\text{states}) = +++-+$.

The following table represents the transition map δ

	$\triangleright$	$\triangleleft$	0	1	2
$q_0^{\rightarrow}$	$q_1^{\rightarrow}/\triangleright$				
$q_1^{\rightarrow}$		$q_2^{\rightarrow}/\triangleleft$	$q_1^{\rightarrow}/0$	$q_1^{\rightarrow}/1$	$q_3^{\leftarrow}/\varepsilon$
$q_2^{\rightarrow}$					
$q_3^{\leftarrow}$	$q_4^{\rightarrow}/1$		$q_4^{\rightarrow}/1$	$q_4^{\rightarrow}/\varepsilon$	$q_4^{\rightarrow}/1$
$q_4^{\rightarrow}$					$q_1^{\rightarrow}/2$

2RFT-Functor Example (cont.)

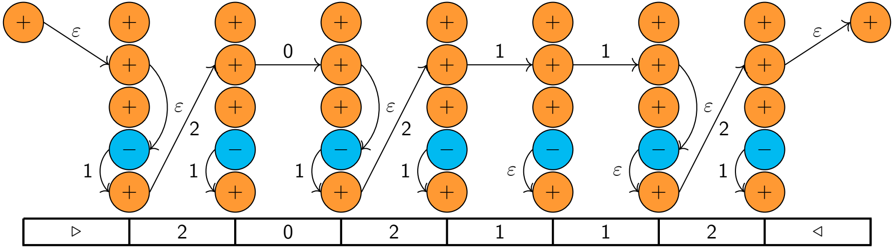
We can read off each column a of the table as the diagram $F(a)$ mapping states to states with the appropriate label on each edge.



So that's our functor.

2RFT-Functor Example (cont.)

If we take a sample run of our transducer on input $w = 202112$ then $F(\triangleright; w; \triangleleft) = F(\triangleright); F(2); F(0); F(2); F(1); F(1); F(2); F(\triangleleft);$.



From which we can read out the composition is merely the single path $+ \to +$ with label 12012112 as expected.

Where do we go from here?

- We are interested not just in these diagrams, but *planar* diagrams.
- Planar diagrams give an autonomous category (not symmetric).
- These planar diagrams give rise to aperiodic transducers.
- Links to the non-commutative linear lambda calculus.

Thank You!
Any Questions?