Containers in Computable Analysis

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Weihrauch Reducibility

Containers

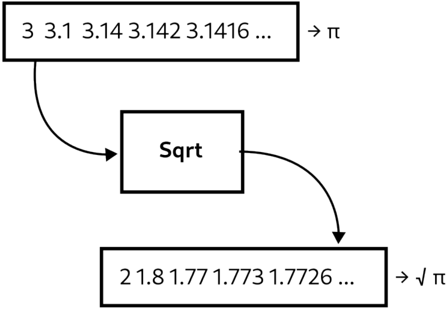
Comparing Structure

Fixpoints

Wrapping Up

Type 2 Computability[1]

- Type 1 machines compute $\mathbb{N} \to \mathbb{N}$
- Type 2 machines compute $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$
- Formally, they are Turing machines which:
  - have an infinite input tape,
  - have a infinite output tape O, and
  - always "make progress":
    $O[n]$ written to, once, in finite time
- A setting for computing with $\mathbb{R}$



[1]Weihrauch (2000) — Computable Analysis: An Introduction

What can we do in $\mathbb{N}^{\mathbb{N}}$?

- A. We can study logical principles that aren't (classically) computable
- Problems = relations on $\mathbb{N}^{\mathbb{N}}$
- BWT: Bolzano-Weierstrass Theorem
- IVT : Intermediate Value Theorem
- LPO: Decide if $w \in \{0,1\}^{\mathbb{N}}$ is constantly 0
- KL: Find an infinite path in an infinite binary tree *given by enumeration*
- Q: How difficult is one *relative* to another?

Reducibility

- A problem A is reducible to a problem B if you can solve A given access to an oracle for B
- Undergraduate computability
  - problem: subsets $A \subseteq \mathbb{N}$
  - solve: give a (polytime) TM for $\chi_A : \mathbb{N} \to \{0, 1\}$
- Reverse Mathematics
  - problem: subsystems of second order arithmetic
  - solve: prove A in $\mathrm{RCA}_0 + B$
- Type 2 computability
  - problem: relation on $\mathbb{N}^{\mathbb{N}}$
  - solve: ?
- Q: How do you even use an oracle in Type 2?

Weihrauch Reducibility: Big Picture

What if you could only make a single oracle call?

```
def problem(arg):
  x = phi(arg)
  res = oracle(x)
  ans = psi(arg, res)
  return ans
```

$$\begin{tikzcd}
\mathrm{arg} \arrow[r, "\varphi"] \arrow[d, dashed, "\mathrm{problem}"] & \varphi(\mathrm{arg}) \arrow[d, dashed, "\mathrm{oracle}"] \\
\mathrm{problem}(\mathrm{arg}) & \mathit{res} \arrow[l, "\psi(\mathrm{arg},\cdot)"]
\end{tikzcd}$$

Definition (Weihrauch Reducible)

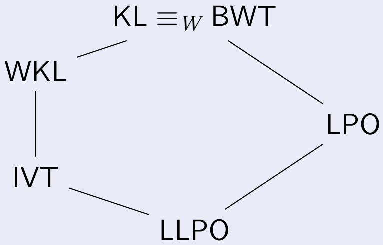
Let f and g be problems. Then f is *Weihrauch reducible*[2] to g, written $f \leq_{\mathrm{W}} g$, if there exist computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\Psi \langle id, g \circ \Phi \rangle \sqsubseteq f$.

[2]Brattka, Gherardi & Pauly (2021) — Weihrauch Complexity in Computable Analysis

Weihrauch Lattice

Proposition (Various[3][4][5])

We have the following separations in the Weihrauch lattice, where WKL *is weak Kőnig's lemma and* LLPO *is the problem* $0 \in \mathrm{LLPO}(p)$ *if* $\forall n \in \mathbb{N}.p(2n) = 0$ *and* $1 \in \mathrm{LLPO}(p)$ *if* $\forall n \in \mathbb{N}.p(2n+1) = 0$*.*



[3]Brattka & Rakotoniaina (2018) — On the Uniform Computational Content of Ramsey's Theorem

[4]Brattka, Gherardi & Marconi (2012) — The Bolzano-Weierstrass Theorem is the Jump of Weak König's Lemma

[5]Brattka & Gherardi (2014) — Effective Choice and Boundedness Principles in Computable Analysis

Example: LPO $\leq_{sW}$ KL

$\varphi : \{0,1\}^{\mathbb{N}} \to$ Binary Trees

0	0	0	1	0	1	...

$\emptyset$

$\psi : \{0,1\}^{\mathbb{N}} \times \mathit{Paths} \to \{0,1\}$

$$\psi(a_n, p_n) = \begin{cases} \mathit{true}, \text{if } p_1 = 1 \\ \mathit{false}, \text{if } p_1 = 0 \end{cases}$$

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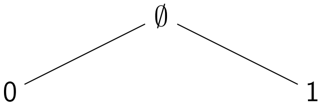

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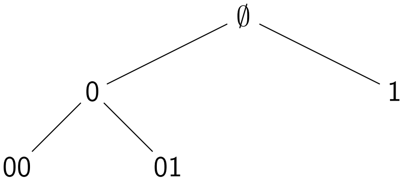

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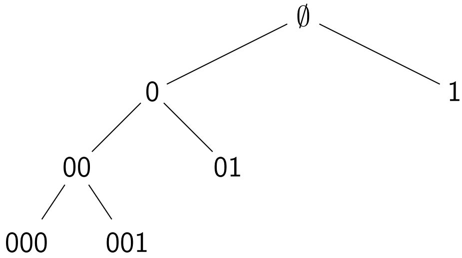

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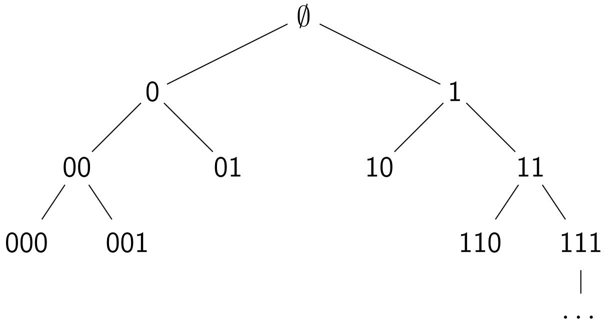

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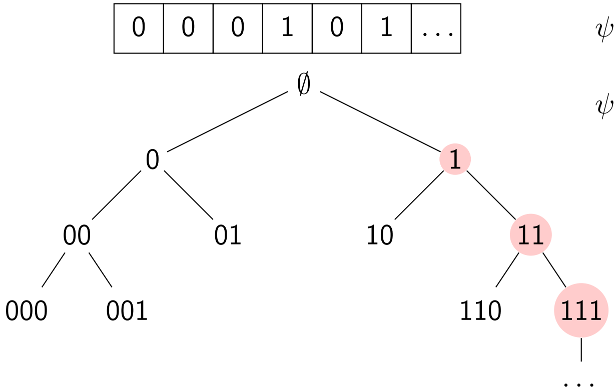

$\psi : \{0,1\}^{\mathbb{N}} \times \mathit{Paths} \to \{0,1\}$

$$\psi(a_n, p_n) = \begin{cases} \mathit{true}, \text{if } p_1 = 1 \\ \mathit{false}, \text{if } p_1 = 0 \end{cases}$$

$\psi(000101\ldots, [1, 11, 111, 111, \ldots]) = \mathit{true}$

Weihrauch Reducibility

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Weihrauch reducibility: redefined

Definition (Problem)

A Weihrauch problem is a family $(F_i)_{i \in I}$ of non-empty subsets of $\mathbb{N}^{\mathbb{N}}$ where $I \subseteq \mathbb{N}^{\mathbb{N}}$.

Definition (Reducible)

$(F_i)_{i \in I}$ is *Weihrauch reducible to* $(G_j)_{j \in J}$ if there exists type 2 computable maps

- $\varphi : I \to J$
- $\forall i \in I$, $\psi(i, \cdot)$ is a map $G_{\varphi(i)} \to F_i$

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This definition doesn't depend very much on the kind of computation!

Families and Bundles

Q: What is the category-theoretic equivalent of a family of sets indexed by a set I?
A: It's maps into I!

$$\begin{aligned} \mathrm{Sets}/I \quad &\simeq \quad \mathrm{Sets}^I \\ f : X \to I \quad &\mapsto \quad \left(f^{-1}(i)\right)_{i \in I} \\ \pi : \coprod_{i \in I} X_i \to I \quad &\leftarrow\!\shortmid \quad (X_i)_{i \in I} \end{aligned}$$

$$\begin{tikzcd} \coprod_{i \in I} G_{\varphi(i)} \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \coprod_{j \in J} G_j \arrow[d, "\pi"] \\ I \arrow[r, "\varphi"] & J \end{tikzcd}$$

Generalising Weihrauch Reducibility via Bundles

Definition (Problem)

A Weihrauch problem in a category $\mathcal{C}$ is a map $X \to I$.

Definition (Reduction)

Given two problems $f : F \to I$ and $g : G \to J$, a *reduction from* $f \to g$ is a pair of maps in $\mathcal{C}$

- $\varphi : I \to J$
- $\psi : G \times_I J \to F$

such that the diagram commutes.

$$\begin{tikzcd}
F \arrow[d, "f"'] & G \times_J I \arrow[l, "\psi"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "g"] \\
I \arrow[r] & I \arrow[r, "\varphi"] & J
\end{tikzcd}$$

This is the definition of container & container morphisms[6]

[6] Abbott, Altenkirch & Ghani (2003) — Categories of Containers

Are all containers Weihrauch problems?

Problem: Problems were defined in terms of families of *non-empty* sets.
Solution: Want the bundles to be *surjective*

Definition (Answerable Containers)

We call a container *answerable* if the underlying map is a pullback-stable epimorphism.

Theorem (Pradic, P.[7])

The Weihrauch degrees are isomorphic to the posetal reflection of the category of answerable containers of represented spaces (rpReprSp).

Represented Spaces = Sets that can be coded in $\mathbb{N}^{\mathbb{N}}$

[7] Pradic & Price (2025) — Weihrauch Problems as Containers

Are all containers Weihrauch problems? (cont.)

Proof Idea.

Containers $\Rightarrow$ Degrees

Let $P : X \to I$ be a container over projective spaces.
Define a problem $\mathrm{w}(P) :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ by:

$$(p, q) \in \mathrm{w}(P) \iff \exists x \in X.\, \delta_I(p) = P(x) \wedge \delta_X(q) = x$$

Degrees $\Rightarrow$ Containers

Let $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a Weihrauch problem.
The projection $\pi_1 : \sum_{p \in \mathrm{dom}(f)} f(p) \to \mathrm{dom}(f)$ is an answerable container. $\square$

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Structure, structure, everywhere

Weihrauch Degrees:

- Poset
- Distributive Lattice
- Parallel Product
- Composition of Degrees
- (In)finite Iterations
- ...

Containers:

- (Bi)Category
- Distributive Category
- Parallel Product
- Composition of Container Functors
- Fixed Points
- ...

Question

To what extent is the structure of the degrees inherited from containers?

Lattice Structure

Proposition

Let $\mathcal{C}$ be a lextensive category and $f : X \to I, g : Y \to J \in \mathrm{Cont}(\mathcal{C})$. Then $\mathrm{Cont}(\mathcal{C})$ is a distributive category with sums $f + g : X + Y \to I + J$ and products $f \times g$ given by $[f \times id_J, id_I \times g] : X \times J + I \times Y \to I \times J$.
Answerable containers are stable under both $+$ and $\times$. 0 is answerable, but 1 is not.

Posetal reflection sends $+$ to $\sqcup$ and $\times$ to $\sqcap$

Corollary

The Weihrauch Degrees form a distributive lattice with a bottom element.

Parallel Products

Proposition

$(\mathrm{Cont}(\mathcal{C}), \otimes, id_1)$ is a monoidal category with $f \otimes g \in \mathrm{Cont}(\mathcal{C})$ given by $f \times g : X \times Y \to I \times J$ in $\mathcal{C}$.

Definition

Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be Weihrauch problems. The parallel product $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$ is given by $(f \times g)(x, z) = f(x) \times g(z)$.

Proposition

Let $f : X \to I$, $g : Y \to J$ be answerable containers. Then $\mathrm{w}(f) \times \mathrm{w}(g) \equiv_W \mathrm{w}(f \otimes g)$.

All the usual properties of $\times$ follow. . .

Composition of Containers

Definition

Let $\mathcal{C}$ be locally cartesian closed. For each container $P : X \to I$, which we will view as an internal family $(X_i)_{i \in I}$, there is a functor $[\![P]\!] : \mathcal{C} \to \mathcal{C}$ defined by $[\![P]\!](Y) = \sum_{i \in I} Y^{X_i}$.

Proposition

There is a monoidal product $\star$ on $\mathrm{Cont}(\mathcal{C})$ *such that* $[\![P \star Q]\!] \cong [\![Q]\!] \circ [\![P]\!]$.

$$P \star Q = \left(\sum_{y \in Y_j} X_{e(y)} \right)_{\substack{j \in J \\ e : Y_j \to I}} \qquad \textit{for all containers } P = (X_i)_{i \in I}, Q = (Y_j)_{j \in J}$$

Composition of Degrees

Definition

Let f, g be Weihrauch problems. The *compositional product*[8] $f \star g$ is defined by

$$(f \star g)\langle x, e\rangle := \{\langle y, z\rangle \mid y \in g(x) \text{ and } z \in f(\Phi(e, y))\}$$

where $\Phi(e, y)$ runs the universal type-2 machine Φ on code e with argument y.

Intuition: x is a question for g, and e is a *strategy* for asking questions to f.
Get back an answer y for x and z for $e(y)$.

Wouldn't it be nice if $\mathrm{w}(f) \star \mathrm{w}(g) \equiv_{\mathrm{W}} \mathrm{w}(f \star g)$?

[8] Westrick (2020) – A note on the diamond operator

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Reality Check

rpReprSp is not (locally) cartesian closed!

[8] Westrick (2020) – A note on the diamond operator

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Non-monoidal operators

Two generalisations of parallel product of degrees:

Definition

Let $f :\subseteq X \rightrightarrows W$ be a problem. Define the following problems.

$$f^* :\subseteq X^* \rightrightarrows Y^* \qquad \text{(finite parallelisation[9])}$$
$$f^*(i, x) := \{i\} \times f(x)$$
$$\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}} \qquad \text{(infinite parallelisation[10])}$$
$$\widehat{f}(x_n)_{n\in\mathbb{N}} := f(x_0) \times f(x_1) \times \ldots$$

Q. How to handle these in a general manner?

[9] Pauly (2010) — How Incomputable is Finding Nash Equilibria?
[10] Brattka (2003) — Computability over Topological Structures

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$$f^*(i, x) := \{i\} \times f(x) \qquad f^* \equiv_{\mathrm{W}} \mathrm{id} \sqcup f \times f^*$$

$$\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}} \qquad \text{(infinite parallelisation[10])}$$

$$\widehat{f}(x_n)_{n \in \mathbb{N}} := f(x_0) \times f(x_1) \times \ldots \qquad \widehat{f} \equiv_{\mathrm{W}} f \times \widehat{f}$$

Q. How to handle these in a general manner?

Observation: both definable by fixpoints of monotone operators

[9] Pauly (2010) — How Incomputable is Finding Nash Equilibria?

[10] Brattka (2003) — Computability over Topological Structures

Fibred Polynomial Functors: Definition

Definition

Let $\mathcal{C}$ be an LCCC. Say that $F : \mathrm{Cont}(\mathcal{C})^k \to \mathrm{Cont}(\mathcal{C})$ is a *fibred polynomial functor* when

- (F, F_0) is fibred functor $\mathrm{shape}^k \to \mathrm{shape}$ for some $F_0 : \mathcal{C}^k \to \mathcal{C}$
- F_0 is a polynomial functor
- for every object $I = (I_1, \ldots, I_k)$ in $\mathcal{C}^k$, the functors on the fibers

$$F_I : \mathcal{C}\big/ I_1 + \ldots + I_k \longrightarrow \mathcal{C}\Big/ F_0(I)$$

induced by the isomorphism $\mathcal{C}/I_1 + \ldots + I_k \cong \mathcal{C}^k/I$ and F are all polynomial functors.

Q. Is this a good definition?

Fibred Polynomial Functors: Fixpoints

Theorem (Pradic, P.[12])

If (F_0, F) is a fibred polynomial endofunctor over containers and $\mathcal{C}$ has dependent M-types and W-types, the following exist:

- *an initial algebra μF for F*
- *a terminal coalgebra νF for F*
- *a (co)algebra ζF for F*

Proof Idea.

Step 1. Take fixpoint $\alpha \in \{\mu, \nu\}$ of F_0.
Step 2. Take fixpoint $\beta \in \{\mu, \nu\}$ of ${}^{C}/_{\alpha}$.
Step 3. Diagram chase shows $(\alpha, \beta) = (\mu, \nu)$ is initial & (ν, μ) is terminal. □

[12] Pradic & Price (2026) – Problems with Fixpoints of Polynomials of Polynomials (Forthcoming)

Fibred Polynomial Functors: Examples

Lemma

The following functors are fibred polynomial:

- *constant functors,*
- *identity and projection functors,*
- *compositions of fibred polynomial functors*
- $\times, +, \otimes : \mathsf{Cont}(\mathcal{C})^2 \to \mathsf{Cont}(\mathcal{C})$,
- $X \mapsto X \star P : \mathsf{Cont}(\mathcal{C})^2 \to \mathsf{Cont}(\mathcal{C})$ *for any container* P

Remark

$X \mapsto P \star X$ *is not fibred in general.*

Operations definable by fixpoints

Notation	Operation on problems	Fixpoint expression
P^*	finite parallelisation	$\mu X.\ \boldsymbol{I} \ + \ P \otimes X$
$\widehat{P}$	infinite parallelisation	$\zeta X.\ P \otimes X$
$\widetilde{P}$	ask ω questions get one answer	$\nu X.\ P \times X$
$P^\diamond$	finite iterations[13] (free monad over a polynomial[14])	$\mu X.\ \boldsymbol{I} + X \star P$
P^∞	ω-loop	$\zeta X.\ X \star P$

[13] Neumann & Pauly (2018) — A topological view on algebraic computation models
[14] Gambino & Kock (2013) — Polynomial Functors and Polynomial Monads

Specific Problems as Fixpoints

Definition (ζ-expressions)

When $\mathcal{C}$ is a $\Pi\mathcal{W}\mathcal{M}$-category, we have the following grammar.

$$E, E' \;::=\; X \mid \mu X.\,E \mid \zeta X.\,E \mid \nu X.\,E \mid E \times E' \mid E \otimes E' \mid E + E'$$

We also make fix the following notations

$$0 := \mu X.X \qquad \boldsymbol{I} := \zeta X.X \qquad 1 := \nu X.X$$

This grammar has an interpretation $\langle\!\langle E \rangle\!\rangle$ in fibred polynomial functors.

Q. Are the natural problems (LPO, WKL, ...) definable as closed expressions?

Specific Problems as Fixpoints (cont)

The bad news:

Proposition

For any closed ζ-expression E, $\langle\!\langle E \rangle\!\rangle \equiv 0$, $\boldsymbol{I}$ or 1.

Now what?

Specific Problems as Fixpoints (cont)

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For any closed ζ-expression E, $\langle\!\langle E \rangle\!\rangle \equiv 0$, $\boldsymbol{I}$ or 1.

Now what?

Definition (Answerable Part)

Let $P : X \to I \in \mathrm{Cont}(\mathrm{ReprSp})$, then $P = \iota \circ \mathrm{Ans}(P)$ for some answerable container $\mathrm{Ans}(P)$ and regular mono ι. We call $\mathrm{Ans}(P)$ the *answerable part of P*.

Proposition

- $\mathrm{LPO} \equiv_W \mathrm{Ans}(\langle\!\langle (\zeta X.\, X + 1) \times (\nu X.\, X + \boldsymbol{I}) \rangle\!\rangle)$
- $\mathrm{WKL} \equiv_W \mathrm{Ans}(\langle\!\langle \zeta X.\, 1 + X \times X \rangle\!\rangle)$

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What we've covered

- Weihrauch reductions are container morphisms
- $\sqcup$, $\sqcap$, $\times$ reflect structure on containers
- Cont(rpReprSp) only a *weak* LCC $\Rightarrow$ weak composition
- Monotone operators on problems $\Rightarrow$ fibred polynomial functors...
- which have fixpoints μ, ν, ζ
- Many natural problems definable using ζ-expressions & Ans

Questions?

Also, you can hire me!
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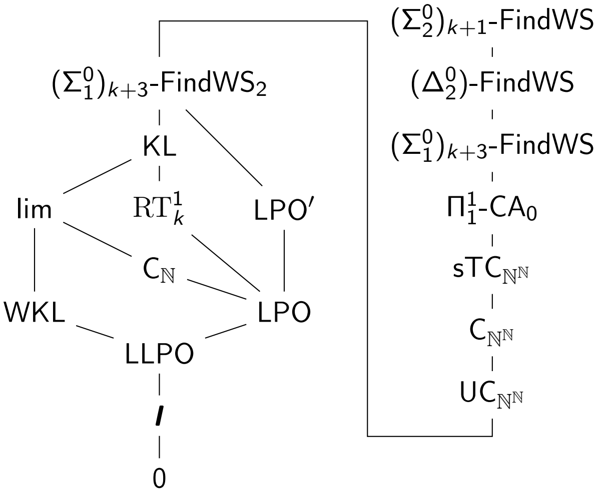
Photo credit: Flickr user mitchgroff (modified) — CC-BY-NC 2.0

The Big Picture



Weihrauch degree D	ζ-expression E with $\mathrm{Ans}(\langle\!\langle E \rangle\!\rangle) \equiv D$
$(\Sigma^0_2)_{2k}$-FindWS	$\zeta X_{2k}.\nu X_{2k-1}.\ldots.\zeta X_1.\nu X_0.\ \widehat{\widetilde{\sum_{i=0}^{2k} X_i}}$
Δ^0_2-FindWS	$\mu Z.\,\zeta X.\ \widehat{\widetilde{X}} + \left(\nu Y.\,\widehat{\widetilde{Y}} + Z\right)$
sT$C_{\mathbb{N}^\mathbb{N}}$	$\left(\nu X.\widehat{X+\mathbf{I}}\right) \times \left(\zeta X.\widetilde{X+1}\right)$
$C_{\mathbb{N}^\mathbb{N}}$	$\zeta X.\,\widetilde{X+1}$
Δ^0_1-FindWS	$\mu X.\,\widehat{\widetilde{X}} + \mathbf{I} + 1$
KL	$\zeta X.\,(\nu Y.\,X+Y) \times (\nu Y.\,X+Y)$
lim	$(\nu X.\,\widehat{X+\mathbf{I}}) \times (\zeta X.\,X+1)$
WKL	$\zeta X.\,1 + X \times X$
RT^1_k	$\underbrace{(\zeta X.\nu Y.\,X+Y) \times \ldots}_{k \text{ times}}$
LPO′	$(\nu X.\,\zeta Y.\,X+Y) \times (\zeta X.\,\nu Y.\,X+Y)$
$C_\mathbb{N}$	$\zeta X.\,\widetilde{X+1}$
LPO	$(\nu X.\,X+\mathbf{I}) \times (\zeta X.\,X+1)$
C_k	$\underbrace{(\zeta X.\,X+1) \times \ldots}_{k \text{ times}}$
$\mathbf{I}$	$\zeta X.\,X$
0	$\mu X.\,X$

KL $\not\leq$ LPO

- Suppose KL $\leq$ LPO and have a weihrauch reduction (φ, ψ).
- $\varphi(t) = 000\ldots$ for some infinite tree t.
  - Otherwise, $\psi(\cdot, \mathit{false})$ implies KL is computable.
- Set $(p_n)_{n\in\mathbb{N}} = \psi(t, \mathit{true})$.
- p_0 will have been output after reading a finite part of t, say t_1.
- $\varphi(t)$ will output 0 after reading a finite part of t, say t_2.
- Any infinite tree that agrees with t on $t_1 \cup t_2$ will output the same p_0.
- So pick one whose only infinite path doesn't start with p_0. $\lightning$

Projectives to the Rescue

- Category of modest sets $\mathsf{Mod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ is an LCCC.
- rpReprSp is the full subcategory of *projectives*
- Every modest set I has a projective cover $\widetilde{I} \twoheadrightarrow I$
- and pulling $P : X \to I$ back along that cover gives $\widetilde{P} : \widetilde{X} \to \widetilde{I}$
- which *is* answerable

Proposition

Let P, Q be answerable containers.

$$\mathrm{w}(\widetilde{P \star Q}) \equiv_W \mathrm{w}(P) \star \mathrm{w}(Q)$$

Q. Do we get all the usual laws "for free"?