Problems with Fixpoints of Polynomials of Polynomials

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Logic in Computer Science
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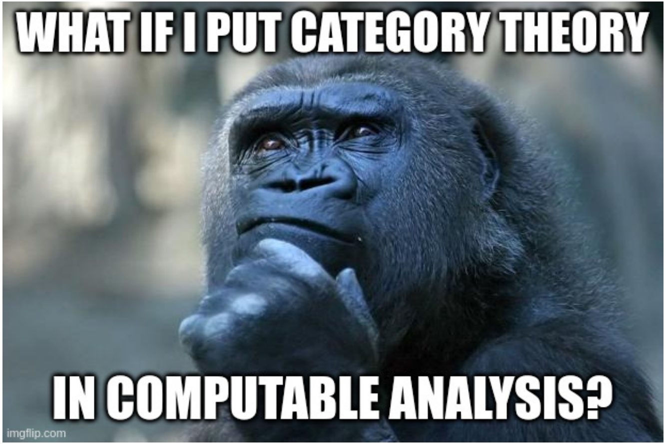
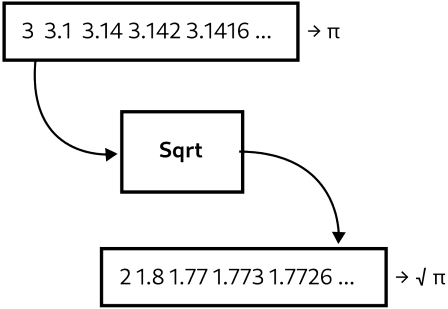
This is Your Brain on Categories



Type 2 Computability[1]

- Type 1 machines compute $\mathbb{N} \to \mathbb{N}$
- Type 2 machines compute $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$
- Formally, they are Turing machines which:
  - have an infinite input tape,
  - have a infinite output tape O, and
  - always "make progress": $O[n]$ written to, once, in finite time
- A setting for computing with $\mathbb{R}$



[1] Weihrauch (2000) — Computable Analysis: An Introduction

What Can We Do in $\mathbb{N}^{\mathbb{N}}$?

- A. We can study logical principles that aren't (classically) computable
- Problems = relations on $\mathbb{N}^{\mathbb{N}}$
- BWT: Bolzano-Weierstrass Theorem
- IVT : Intermediate Value Theorem
- LPO: Decide if $w \in \{0,1\}^{\mathbb{N}}$ is constantly 0
- KL: Find an infinite path in an infinite binary tree *given by enumeration*
- Q: How difficult is one *relative* to another?

Reducibility

- A problem A is reducible to a problem B if you can solve A given access to an oracle for B
- Undergraduate computability
  - problem: subsets $A \subseteq \mathbb{N}$
  - solve: give a (polytime) TM for $\chi_A : \mathbb{N} \to \{0, 1\}$
- Reverse Mathematics
  - problem: subsystems of second order arithmetic
  - solve: prove A in $\mathrm{RCA}_0 + B$
- Type 2 computability
  - problem: relation on $\mathbb{N}^{\mathbb{N}}$
  - solve: ?
- Q: How do you even use an oracle in Type 2?

Weihrauch Reducibility: Big Picture

What if you could only make a single oracle call?

```python
def problem(arg):
  x = phi(arg)
  res = oracle(x)
  ans = psi(arg, res)
  return ans
```

$$\begin{tikzcd}
\mathrm{arg} \arrow[r, "\varphi"] \arrow[d, dashed, "\mathrm{problem}"] & \varphi(\mathrm{arg}) \arrow[d, dashed, "\mathrm{oracle}"] \\
\mathrm{problem}(\mathrm{arg}) & \mathit{res} \arrow[l, "\psi(\mathrm{arg}, \cdot)"]
\end{tikzcd}$$

Definition (Weihrauch Reducible)

Let f and g be problems. Then f is *Weihrauch reducible*[2] to g, written $f \leq_{\mathrm{W}} g$, if there exist computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\Psi \langle id, g \circ \Phi \rangle \sqsubseteq f$.

[2] Brattka, Gherardi & Pauly (2021) — Weihrauch Complexity in Computable Analysis

Example: LPO $\leq_{sW}$ KL

$\varphi : \{0,1\}^{\mathbb{N}} \to$ Binary Trees

0	0	0	1	0	1	...

$\emptyset$

$\psi : \{0,1\}^{\mathbb{N}} \times \mathit{Paths} \to \{0,1\}$

$$\psi(a_n, p_n) = \begin{cases} \mathit{true}, \text{if } p_1 = 1 \\ \mathit{false}, \text{if } p_1 = 0 \end{cases}$$

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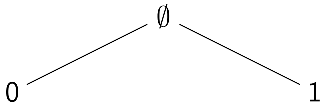

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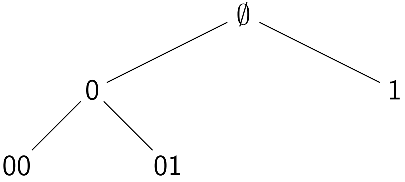

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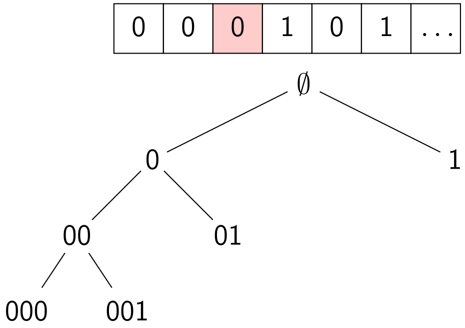

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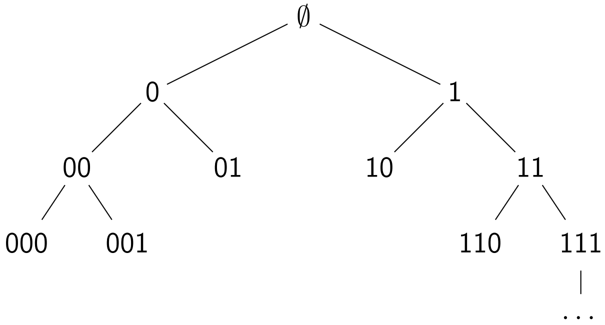

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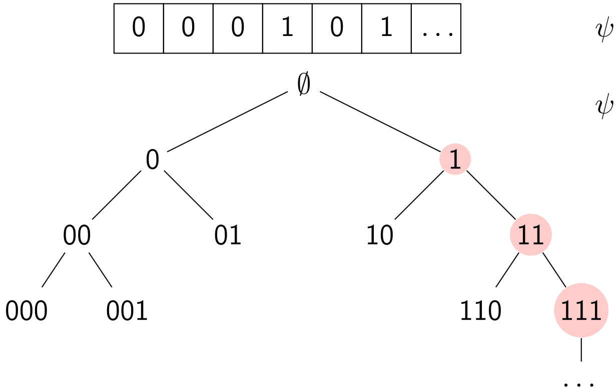

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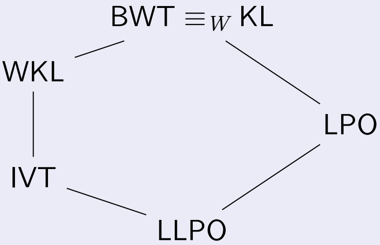
$$\psi(a_n, p_n) = \begin{cases} \mathit{true}, \text{if } p_1 = 1 \\ \mathit{false}, \text{if } p_1 = 0 \end{cases}$$

$$\psi(000101\ldots, [1, 11, 111, 111, \ldots]) = \mathit{true}$$

Weihrauch Lattice

Proposition (Various[3][4][5])

We have the following separations in the Weihrauch lattice, where WKL *is weak Kőnig's lemma and* LLPO *is the problem* $0 \in \mathrm{LLPO}(p)$ *if* $\forall n \in \mathbb{N}. p(2n) = 0$ *and* $1 \in \mathrm{LLPO}(p)$ *if* $\forall n \in \mathbb{N}. p(2n+1) = 0$.



[3] Brattka & Rakotoniaina (2018) — On the Uniform Computational Content of Ramsey's Theorem

[4] Brattka, Gherardi & Marconi (2012) — The Bolzano-Weierstrass Theorem is the Jump of Weak König's Lemma

[5] Brattka & Gherardi (2014) — Effective Choice and Boundedness Principles in Computable Analysis

Weihrauch Reducibility: Redefined

Definition (Problem)

A Weihrauch problem is a family $(F_i)_{i \in I}$ of non-empty subsets of $\mathbb{N}^{\mathbb{N}}$ where $I \subseteq \mathbb{N}^{\mathbb{N}}$.

Definition (Reducible)

$(F_i)_{i \in I}$ is *Weihrauch reducible to* $(G_j)_{j \in J}$ if there exists type 2 computable maps

- $\varphi : I \to J$
- $\forall i \in I$, $\psi(i, \cdot)$ is a map $G_{\varphi(i)} \to F_i$

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This definition doesn't depend very much on the kind of computation!

Generalising Weihrauch Reducibility via Bundles

Definition (Problem)

A Weihrauch problem in a category $\mathcal{C}$ is a map $X \to I$.

Definition (Reduction)

Given two problems $f : F \to I$ and $g : G \to J$, a *reduction from* $f \to g$ is a pair of maps in $\mathcal{C}$

- $\varphi : I \to J$
- $\psi : G \times_I J \to F$

$$\begin{tikzcd} F \arrow[d, "f"'] & G \times_J I \arrow[l, "\psi"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "g"] \\ I \arrow[r, equal] & I \arrow[r, "\varphi"] & J \end{tikzcd}$$

such that the diagram commutes.

This is the definition of container & container morphisms[6]

[6] Abbott, Altenkirch & Ghani (2003) — Categories of Containers

Are *All* Containers Weihrauch problems?

Problem: Problems were defined in terms of families of *non-empty* sets.
Solution: Want the containers to be *surjective*

Definition (Answerable Containers)

We call a container *answerable* if the underlying map is a pullback-stable epimorphism.

Theorem (Pradic, P.[7])

The Weihrauch degrees are isomorphic to the posetal reflection of the category of answerable containers of represented spaces (rpReprSp).

Represented Spaces = Sets that can be coded in $\mathbb{N}^{\mathbb{N}}$

[7] Pradic & Price (2025) — Weihrauch Problems as Containers

Containers & Polynomial Functors

Definition

Let $\mathcal{C}$ be locally cartesian closed. For each container $P : X \to I$, which we will view as an internal family $(X_i)_{i \in I}$, there is a functor $[\![P]\!] : \mathcal{C} \to \mathcal{C}$ defined by $[\![P]\!](Y) = \sum_{i \in I} Y^{X_i}$.

Proposition (Facts)

- *There is a monoidal product* $\star$ *on* $\mathrm{Cont}(\mathcal{C})$ *such that* $[\![P \star Q]\!] \cong [\![Q]\!] \circ [\![P]\!]$.
- $\mathrm{Cont}(\mathcal{C}) \simeq \mathrm{Poly}(\mathcal{C})$

Caution!

rpReprSp is not (locally) cartesian closed!

Structure, Structure, Everywhere

Weihrauch Degrees:

- Poset
- Distributive Lattice
- Parallel Product
- Composition of Degrees
- (In)finite Iterations
- ...

Containers:

- (Bi)Category
- Distributive Category
- Parallel Product
- Composition $\star$ of Containers
- Fixed Points
- ...

Question

To what extent is the structure of the degrees inherited from containers?

Example: Parallel Products

Proposition

$(\mathrm{Cont}(\mathcal{C}), \otimes, id_1)$ is a monoidal category with $f \otimes g \in \mathrm{Cont}(\mathcal{C})$ given by $f \times g : X \times Y \to I \times J$ in $\mathcal{C}$.

Definition

Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be Weihrauch problems. The parallel product $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$ is given by $(f \times g)(x, z) = f(x) \times g(z)$.

Proposition

Let $f : X \to I$, $g : Y \to J$ be answerable containers. Then $\mathrm{w}(f) \times \mathrm{w}(g) \equiv_W \mathrm{w}(f \otimes g)$.

All the usual properties of $\times$ follow...

Non-Monoidal Operators : Generalising Parallel Products

Definition

Let $f :\subseteq X \rightrightarrows W$ be a problem. Define the following problems.

$$f^* :\subseteq X^* \rightrightarrows Y^* \qquad \text{(finite parallelisation}^{8}\text{)}$$
$$f^*(i, x) := \{i\} \times f(x)$$
$$\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}} \qquad \text{(infinite parallelisation}^{9}\text{)}$$
$$\widehat{f}(x_n)_{n\in\mathbb{N}} := f(x_0) \times f(x_1) \times \ldots$$

Q. How to handle these in a general manner?

[8] Pauly (2010) — How Incomputable is Finding Nash Equilibria?
[9] Brattka (2003) — Computability over Topological Structures

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$$f^* :\subseteq X^* \rightrightarrows Y^* \qquad \text{(finite parallelisation[8])}$$

$$f^*(i, x) := \{i\} \times f(x) \qquad f^* \equiv_{\mathrm{W}} \mathrm{id} \sqcup f \times f^*$$

$$\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}} \qquad \text{(infinite parallelisation[9])}$$

$$\widehat{f}(x_n)_{n\in\mathbb{N}} := f(x_0) \times f(x_1) \times \ldots \qquad \widehat{f} \equiv_{\mathrm{W}} f \times \widehat{f}$$

Q. How to handle these in a general manner?

Observation: both definable by fixpoints of "polynomial" monotone operators

[8] Pauly (2010) — How Incomputable is Finding Nash Equilibria?

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A Naïve Approach

So just use polynomial functors $\mathsf{Cont}(\mathcal{C}) \to \mathsf{Cont}(\mathcal{C})$?

Um, Actually...

$\mathsf{Cont}(\mathcal{C})$ isn't locally cartesian closed[10]

Besides, the "usual" theory only gives initial algebras / terminal coalgebras...

and $\widehat{f}$ is neither a least or greatest fixpoint!

[10] Altenkirch, Levy & Staton – Higher Order Containers

Fibred Polynomial Functors: Definition

Definition

Let $\mathcal{C}$ be an LCCC. Say that $F : \mathrm{Cont}(\mathcal{C})^k \to \mathrm{Cont}(\mathcal{C})$ is a *fibred polynomial functor* when

- (F, F_0) is fibred functor $\mathrm{shape}^k \to \mathrm{shape}$ for some $F_0 : \mathcal{C}^k \to \mathcal{C}$
- F_0 is a polynomial functor
- for every object $I = (I_1, \ldots, I_k)$ in $\mathcal{C}^k$, the functors on the fibers

  $$F_I : {}^{\mathcal{C}}\!\big/_{I_1 + \ldots + I_k} \longrightarrow {}^{\mathcal{C}}\!\big/_{F_0(I)}$$

  induced by the isomorphism ${}^{\mathcal{C}}/_{I_1 + \ldots + I_k} \cong {}^{\mathcal{C}^k}/_{I}$ and F are all polynomial functors.

Q. Is this a good definition?

Fibred Polynomial Functors: Fixpoints

Theorem (Pradic, P.)

If (F_0, F) is a fibred polynomial endofunctor over containers and $\mathcal{C}$ has dependent M-types and W-types, the following fixpoints exist:

- *an initial algebra μF for F*
- *a terminal coalgebra νF for F*
- *a (co)algebra ζF for F*

Proof Idea.

Step 1. Take fixpoint $\alpha \in \{\mu, \nu\}$ of F_0.
Step 2. Take fixpoint $\beta \in \{\mu, \nu\}$ of ${}^{C}/_{\alpha}$.
Step 3. Diagram chase shows $(\alpha, \beta) = (\mu, \nu)$ is initial & (ν, μ) is terminal. $\square$

It turns out (μ, μ) is also initial!

Fibred Polynomial Functors: Examples

Lemma

The following functors are fibred polynomial:

- *constant functors,*
- *identity and projection functors,*
- *compositions of fibred polynomial functors*
- $\times, +, \otimes : \mathsf{Cont}(\mathcal{C})^2 \to \mathsf{Cont}(\mathcal{C})$,
- $X \mapsto X \star P : \mathsf{Cont}(\mathcal{C})^2 \to \mathsf{Cont}(\mathcal{C})$ *for any container* P

Remark

$X \mapsto P \star X$ *is not fibred in general.*

Operations Definable by Fixpoints

Notation	Operation on problems	Fixpoint expression
P^*	finite parallelisation	$\mu X.\ \boldsymbol{I} + P \otimes X$
$\widehat{P}$	infinite parallelisation	$\zeta X.\ P \otimes X$
$\widetilde{P}$	ask ω questions get one answer	$\nu X.\ P \times X$
$P^\diamond$	finite iterations[11] (free monad over a polynomial[12])	$\mu X.\ \boldsymbol{I} + X \star P$
P^∞	ω-loop	$\zeta X.\ X \star P$

[11] Neumann & Pauly (2018) — A topological view on algebraic computation models
[12] Gambino & Kock (2013) — Polynomial Functors and Polynomial Monads

Specific Problems as Fixpoints

Definition (ζ-expressions)

When $\mathcal{C}$ is a $\Pi\mathcal{W}\mathcal{M}$-category, we have the following grammar.

$$E, E' \ ::= \ X \mid \mu X.\, E \mid \zeta X.\, E \mid \nu X.\, E \mid E \times E' \mid E \otimes E' \mid E + E'$$

We also fix the following notations

$$0 := \mu X.X \qquad \boldsymbol{I} := \zeta X.X \qquad 1 := \nu X.X$$

This grammar has an interpretation $\langle\!\langle E \rangle\!\rangle$ in fibred polynomial functors.

And $\langle\!\langle E \rangle\!\rangle$ restricts to a functor $\mathsf{Cont}(\mathsf{rpReprSp})^{|\chi|} \to \mathsf{Cont}(\mathsf{rpReprSp})$.

Q. Are the natural problems $(\mathsf{LPO}, \mathsf{WKL}, \ldots)$ definable as closed expressions?

Specific Problems as Fixpoints (cont.)

The bad news:

Proposition

For any closed ζ-expression E, $\langle\!\langle E \rangle\!\rangle \equiv_W 0$, $\boldsymbol{I}$ or 1.

Now what?

Specific Problems as Fixpoints (cont.)

The bad news:

Proposition

For any closed ζ-expression E, $\langle\!\langle E \rangle\!\rangle \equiv_W 0$, $\boldsymbol{I}$ or 1.

Now what? Throw out the junk!

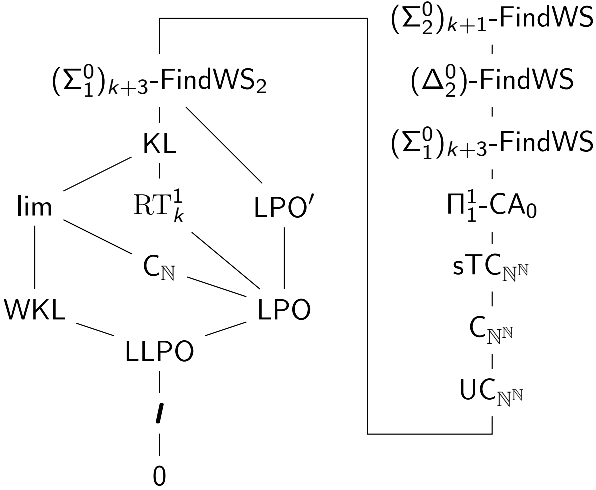
Definition (Answerable Part)

Let $P : X \to I \in \mathrm{Cont}(\mathrm{ReprSp})$, then $P = \iota \circ \mathrm{Ans}(P)$ for some answerable container $\mathrm{Ans}(P)$ and regular mono ι. We call $\mathrm{Ans}(P)$ the *answerable part of P*.

Proposition

- $\mathrm{LPO} \equiv_W \mathrm{Ans}(\langle\!\langle (\zeta X.\ X + 1) \times (\nu X.\ X + \boldsymbol{I}) \rangle\!\rangle)$
- $\mathrm{WKL} \equiv_W \mathrm{Ans}(\langle\!\langle \zeta X.\ 1 + X \times X \rangle\!\rangle)$

The Big Picture



Weihrauch degree D	ζ-expression E with $\mathrm{Ans}(\langle\!\langle E \rangle\!\rangle) \equiv D$
$(\Sigma^0_2)_{2k}$-FindWS	$\zeta X_{2k}.\nu X_{2k-1}.\dots.\zeta X_1.\nu X_0.\ \widehat{\widetilde{\sum_{i=0}^{2k} X_i}}$
Δ^0_2-FindWS	$\mu Z.\ \zeta X.\ \hat{\hat{\tilde{X}}} + \left(\nu Y.\ \hat{\tilde{Y}} + Z\right)$
sTC$_{\mathbb{N}^\mathbb{N}}$	$\left(\nu X.\widehat{X+\boldsymbol{I}}\right) \times \left(\zeta X.\widetilde{X+1}\right)$
$\mathrm{C}_{\mathbb{N}^\mathbb{N}}$	$\zeta X.\ \widetilde{X+1}$
Δ^0_1-FindWS	$\mu X.\ \hat{\hat{\tilde{X}}} + \boldsymbol{I} + 1$
KL	$\zeta X.\ (\nu Y.\ X+Y) \times (\nu Y.\ X+Y)$
lim	$(\nu X.\ X+\boldsymbol{I}) \times (\zeta X.\ X+1)$
WKL	$\zeta X.\ 1 + X \times X$
RT^1_k	$\underbrace{(\zeta X.\nu Y.\ X+Y) \times \dots}_{k \text{ times}}$
LPO′	$(\nu X.\ \zeta Y.\ X+Y) \times (\zeta X.\ \nu Y.\ X+Y)$
$\mathrm{C}_{\mathbb{N}}$	$\zeta X.\ \widetilde{X+1}$
LPO	$(\nu X.\ X+\boldsymbol{I}) \times (\zeta X.\ X+1)$
C_k	$\underbrace{(\zeta X.\ X+1) \times \dots}_{k \text{ times}}$
$\boldsymbol{I}$	$\zeta X.\ X$
0	$\mu X.\ X$

Questions?

Also, I am looking for a job!
ian@countingishard.org

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