# Weihrauch Reducibility from a Categorical Lens

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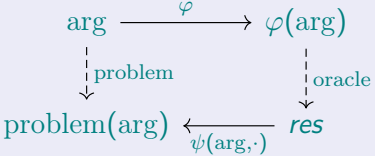
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# Weihrauch Reducibility: Big Picture

- We want to compare the difficulty of problems in computable analysis
- Problems = partial multivalued functions on $\mathbb{N}^{\mathbb{N}}$
- Using an oracle seems problematic... maybe just one use will do

## Definition (Weihrauch Reducible)

Let $f$ and $g$ be problems. Then $f$ is *Weihrauch reducible*[1] to $g$, written $f \leq_{\mathrm{W}} g$, if there exist computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\Psi \langle id, g \circ \Phi \rangle \sqsubseteq f$.




[1] Brattka, Gherardi & Pauly (2021) — Weihrauch Complexity in Computable Analysis

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# Weihrauch Reducibility: Redefined

## Definition (Problem)

A Weihrauch problem is a family $(F_i)_{i \in I}$ of non-empty subsets of $\mathbb{N}^{\mathbb{N}}$ where $I \subseteq \mathbb{N}^{\mathbb{N}}$.

## Definition (Reducible)

$(F_i)_{i \in I}$ is *Weihrauch reducible to* $(G_j)_{j \in J}$ if there exists type 2 computable maps

- $\varphi : I \to J$
- $\forall i \in I$, $\psi(i, \cdot)$ is a map $G_{\varphi(i)} \to F_i$

# Weihrauch Reducibility: Redefined

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This definition doesn't depend very much on the kind of computation!

# Families and Bundles

Q: What is the category-theoretic equivalent of a family of sets indexed by a set $I$?
A: It's maps into $I$!

$$\begin{array}{rcl} \mathbf{Sets}/I & \simeq & \mathbf{Sets}^I \\ f : X \to I & \mapsto & \left(f^{-1}(i)\right)_{i \in I} \\ \pi : \bigsqcup_{i \in I} X_i \to I & \leftarrow\!\shortmid & (X_i)_{i \in I} \end{array}$$

$$\begin{array}{ccc} \bigsqcup_{i \in I} G_{\varphi(i)} & \longrightarrow & \bigsqcup_{j \in J} G_j \\ \downarrow{\scriptstyle \pi} & \lrcorner & \downarrow{\scriptstyle \pi} \\ I & \xrightarrow{\varphi} & J \end{array}$$

# Generalising Weihrauch Reducibility via Bundles

## Definition (Problem)

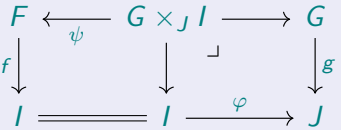
A Weihrauch problem in a category $\mathcal{C}$ is a map $X \to I$.

## Definition (Reduction)

Given two problems $f : F \to I$ and $g : G \to J$, a *reduction from* $f \to g$ is a pair of maps in $\mathcal{C}$

- $\varphi : I \to J$
- $\psi : G \times_I J \to F$

such that the diagram commutes.




This is the definition of container & container morphisms[2]

[2] Abbott, Altenkirch & Ghani (2003) — Categories of Containers

# Are *All* Containers Weihrauch problems?

Problem: Problems were defined in terms of families of *non-empty* sets.
Solution: Want the containers to be *surjective*

**Definition (Answerable Containers)**

We call a container *answerable* if the underlying map is a pullback-stable epimorphism.

**Theorem (Pradic, P.[3])**

*The Weihrauch degrees are isomorphic to the posetal reflection of the category of answerable containers of represented spaces (rpReprSp).*

Represented Spaces = Sets that can be coded in $\mathbb{N}^{\mathbb{N}}$

[3] Pradic & Price (2025) — Weihrauch Problems as Containers

# Are all containers Weihrauch problems? (cont.)

**Proof Idea.**

**Containers $\Rightarrow$ Degrees**

Let $P : X \to I$ be a container over projective spaces.

Define a problem $\mathrm{w}(P) :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ by:

$$(p, q) \in \mathrm{w}(P) \iff \exists x \in X.\, \delta_I(p) = P(x) \wedge \delta_X(q) = x$$

**Degrees $\Rightarrow$ Containers**

Let $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a Weihrauch problem.

The projection $\pi_1 : \sum_{p \in \mathrm{dom}(f)} f(p) \to \mathrm{dom}(f)$ is an answerable container. □

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# Structure, Structure, Everywhere

Weihrauch Degrees:

- Poset
- Distributive Lattice
- Parallel Product
- Composition of Degrees
- (In)finite Iterations
- ...

Containers:

- (Bi)Category
- Distributive Category
- Parallel Product
- Composition $\star$ of Containers
- Fixed Points
- ...

**Question**

To what extent is the structure of the degrees inherited from containers?

# Lattice Structure

## Proposition

*Let $\mathcal{C}$ be a lextensive category and $f : X \to I, g : Y \to J \in \mathrm{Cont}(\mathcal{C})$. Then $\mathrm{Cont}(\mathcal{C})$ is a distributive category with sums $f + g : X + Y \to I + J$ and products $f \times g$ given by $[f \times id_J, id_I \times g] : X \times J + I \times Y \to I \times J$.*
*Answerable containers are stable under both $+$ and $\times$. $0$ is answerable, but $1$ is not.*

Posetal reflection sends $+$ to $\sqcup$ and $\times$ to $\sqcap$

## Corollary

*The Weihrauch Degrees form a distributive lattice with a bottom element.*

# Example: Parallel Products

**Proposition**

*$(\mathsf{Cont}(\mathcal{C}), \otimes, id_1)$ is a monoidal category with $f \otimes g \in \mathsf{Cont}(\mathcal{C})$ given by $f \times g : X \times Y \to I \times J$ in $\mathcal{C}$.*

**Definition**

Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be Weihrauch problems. The parallel product $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$ is given by $(f \times g)(x, z) = f(x) \times g(z)$.

**Proposition**

*Let $f : X \to I$, $g : Y \to J$ be answerable containers. Then $\mathsf{w}(f) \times \mathsf{w}(g) \equiv_W \mathsf{w}(f \otimes g)$.*

All the usual properties of $\times$ follow...

# Composition of Containers

## Definition

Let $\mathcal{C}$ be locally cartesian closed. For each container $P : X \to I$, which we will view as an internal family $(X_i)_{i \in I}$, there is a functor $[\![P]\!] : \mathcal{C} \to \mathcal{C}$ defined by $[\![P]\!](Y) = \sum_{i \in I} Y^{X_i}$.

## Proposition

*There is a monoidal product* $\star$ *on* $\mathsf{Cont}(\mathcal{C})$ *such that* $[\![P \star Q]\!] \cong [\![Q]\!] \circ [\![P]\!]$.

$$P \star Q = \left( \sum_{y \in Y_j} X_{e(y)} \right)_{\substack{j \in J \\ e : Y_j \to I}} \quad \textit{for all containers } P = (X_i)_{i \in I}, Q = (Y_j)_{j \in J}$$

# Composition of Degrees

## Definition

Let $f, g$ be Weihrauch problems. The *compositional product*[4] $f \star g$ is defined by

$$(f \star g)\langle x, e\rangle := \{\langle y, z\rangle \mid y \in g(x) \text{ and } z \in f(\Phi(e, y))\}$$

where $\Phi(e, y)$ runs the universal type-2 machine $\Phi$ on code $e$ with argument $y$.

Intuition: $x$ is a question for $g$, and $e$ is a *strategy* for asking questions to $g$.
Get back an answer $y$ for $x$ and $z$ for $e(y)$.

Wouldn't it be nice if $\mathrm{w}(f) \star \mathrm{w}(g) \equiv_{\mathrm{W}} \mathrm{w}(f \star g)$?

[4] Westrick (2020) – A note on the diamond operator

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## Reality Check

rpReprSp is not (locally) cartesian closed!

[4] Westrick (2020) – A note on the diamond operator

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# Non-Monoidal Operators : Generalising Parallel Products

## Definition

Let $f :\subseteq X \rightrightarrows W$ be a problem. Define the following problems.

$$f^* :\subseteq X^* \rightrightarrows Y^* \qquad \text{(finite parallelisation[5])}$$

$$f^*(i, x) := \{i\} \times f(x)$$

$$\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}} \qquad \text{(infinite parallelisation[6])}$$

$$\widehat{f}(x_n)_{n\in\mathbb{N}} := f(x_0) \times f(x_1) \times \dots$$

Q. How to handle these in a general manner?

[5] Pauly (2010) — How Incomputable is Finding Nash Equilibria?
[6] Brattka (2003) — Computability over Topological Structures

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$$f^*(i, x) := \{i\} \times f(x) \qquad f^* \equiv_{\mathrm{W}} \mathrm{id} \sqcup f \times f^*$$

$$\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}} \qquad \text{(infinite parallelisation[6])}$$

$$\widehat{f}(x_n)_{n\in\mathbb{N}} := f(x_0) \times f(x_1) \times \ldots \qquad \widehat{f} \equiv_{\mathrm{W}} f \times \widehat{f}$$

Q. How to handle these in a general manner?
Observation: both definable by fixpoints of "polynomial" monotone operators

[5] Pauly (2010) — How Incomputable is Finding Nash Equilibria?
[6] Brattka (2003) — Computability over Topological Structures

# A Naïve Approach

So just use polynomial functors $\mathsf{Cont}(\mathcal{C}) \to \mathsf{Cont}(\mathcal{C})$?

**Um, Actually...**

$\mathsf{Cont}(\mathcal{C})$ isn't locally cartesian closed[7]

Besides, the "usual" theory only gives initial algebras / terminal coalgebras...

and $\widehat{f}$ is neither a least or greatest fixpoint!

[7] Altenkirch, Levy & Staton – Higher Order Containers

# Fibred Polynomial Functors: Definition

## Definition (Pradic, P.[8])

Let $\mathcal{C}$ be an LCCC. Say that $F : \mathrm{Cont}(\mathcal{C})^k \to \mathrm{Cont}(\mathcal{C})$ is a *fibred polynomial functor* when

- $(F, F_0)$ is fibred functor $\mathrm{shape}^k \to \mathrm{shape}$ for some $F_0 : \mathcal{C}^k \to \mathcal{C}$
- $F_0$ is a polynomial functor
- for every object $I = (I_1, \ldots, I_k)$ in $\mathcal{C}^k$, the functors on the fibers

$$F_I : \mathcal{C}/I_1 + \ldots + I_k \longrightarrow \mathcal{C}\Big/F_0(I)$$

induced by the isomorphism $\mathcal{C}/I_1 + \ldots + I_k \cong \mathcal{C}^k/I$ and $F$ are all polynomial functors.

Q. Is this a good definition?

[8] Pradic & Price (2026) — Problems with Fixpoints of Polynomials of Polynomials

# Fibred Polynomial Functors: Fixpoints

**Theorem**

*If $(F_0, F)$ is a fibred polynomial endofunctor and $\mathcal{C}$ is an LCCC with dependent M-types and W-types, the following fixpoints exist:*

- *an initial algebra $\mu F$ for $F$*
- *a terminal coalgebra $\nu F$ for $F$*
- *a (co)algebra $\zeta F$ for $F$*

**Proof Idea.**

**Step 1.** Take fixpoint $\alpha \in \{\mu, \nu\}$ of $F_0$.
**Step 2.** Take fixpoint $\beta \in \{\mu, \nu\}$ of $\mathcal{C}/_{\alpha}$.
**Step 3.** Diagram chase shows $(\alpha, \beta) = (\mu, \nu)$ is initial & $(\nu, \mu)$ is terminal. □

It turns out $(\mu, \mu)$ is also initial!

# Fibred Polynomial Functors: Examples

**Lemma**

*The following functors are fibred polynomial:*

- *constant functors,*
- *identity and projection functors,*
- *compositions of fibred polynomial functors*
- $\times, +, \otimes : \mathrm{Cont}(\mathcal{C})^2 \to \mathrm{Cont}(\mathcal{C})$,
- $X \mapsto X \star P : \mathrm{Cont}(\mathcal{C})^2 \to \mathrm{Cont}(\mathcal{C})$ *for any container* $P$

**Remark**

$X \mapsto P \star X$ *is not fibred in general.*

# Operations Definable by Fixpoints

| Notation | Operation on problems | Fixpoint expression |
|---|---|---|
| $P^*$ | finite parallelisation | $\mu X.\ \boldsymbol{I} + P \otimes X$ |
| $\widehat{P}$ | infinite parallelisation | $\zeta X.\ P \otimes X$ |
| $\widetilde{P}$ | ask $\omega$ questions<br>get one answer | $\nu X.\ P \times X$ |
| $P^\diamond$ | finite iterations[9]<br>(free monad over a polynomial[10]) | $\mu X.\ \boldsymbol{I} + X \star P$ |
| $P^\infty$ | $\omega$-loop / inverse limit | $\zeta X.\ X \star P$ |

[9] Neumann & Pauly (2018) — A topological view on algebraic computation models
[10] Gambino & Kock (2013) — Polynomial Functors and Polynomial Monads

# Specific Problems as Fixpoints

## Definition ($\zeta$-expressions)

When $\mathcal{C}$ is a $\Pi\mathcal{WM}$-category, we have the following grammar.

$$E, E' \ ::= \ X \mid \mu X.\, E \mid \zeta X.\, E \mid \nu X.\, E \mid E \times E' \mid E \otimes E' \mid E + E'$$

We also fix the following notations

$$0 := \mu X.X \qquad \boldsymbol{I} := \zeta X.X \qquad 1 := \nu X.X$$

This grammar has an interpretation $\langle\!\langle E \rangle\!\rangle$ in fibred polynomial functors.

And $\langle\!\langle E \rangle\!\rangle$ restricts to a functor $\mathsf{Cont}(\mathsf{rpReprSp})^{|\chi|} \to \mathsf{Cont}(\mathsf{rpReprSp})$.

Q. Are the natural problems ($\mathsf{LPO}, \mathsf{WKL}, \ldots$) definable as closed expressions?

# Specific Problems as Fixpoints (cont.)

The bad news:

**Proposition**

*For any closed $\zeta$-expression $E$, $\langle\!\langle E \rangle\!\rangle \equiv_W 0$, $I$ or $1$.*

Now what?

# Specific Problems as Fixpoints (cont.)

The bad news:

**Proposition**

*For any closed $\zeta$-expression $E$, $\langle\!\langle E \rangle\!\rangle \equiv_W 0$, $\boldsymbol{I}$ or $1$.*

Now what? Throw out the junk!

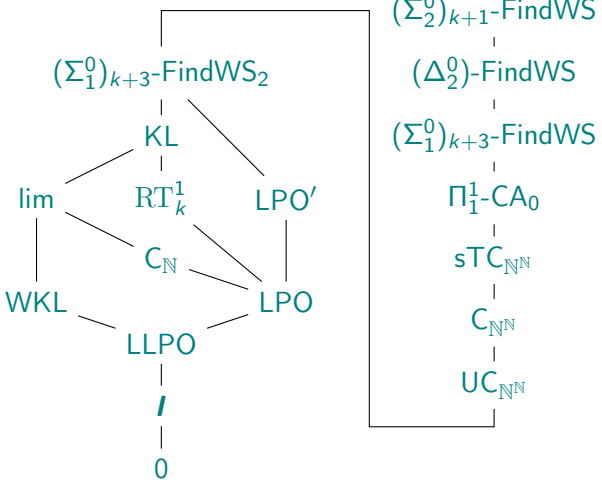
**Definition (Answerable Part)**

Let $P : X \to I \in \mathsf{Cont}(\mathsf{ReprSp})$, then $P = \iota \circ \mathsf{Ans}(P)$ for some answerable container $\mathsf{Ans}(P)$ and regular mono $\iota$. We call $\mathsf{Ans}(P)$ the *answerable part of* $P$.

**Proposition**

- $\mathrm{LPO} \equiv_W \mathsf{Ans}(\langle\!\langle (\zeta X.\, X + 1) \times (\nu X.\, X + \boldsymbol{I}) \rangle\!\rangle)$
- $\mathrm{WKL} \equiv_W \mathsf{Ans}(\langle\!\langle \zeta X.\, 1 + X \times X \rangle\!\rangle)$

# This is not a cheap trick!




| Weihrauch degree $D$ | $\zeta$-expression $E$ with $\mathrm{Ans}(\langle\!\langle E \rangle\!\rangle) \equiv D$ |
|---|---|
| $(\Sigma^0_2)_{2k}$-FindWS | $\zeta X_{2k}.\nu X_{2k-1}.\ldots\zeta X_1.\nu X_0.\ \widehat{\sum_{i=0}^{2k} X_i}$ |
| $\Delta^0_2$-FindWS | $\mu Z.\,\zeta X.\ \widehat{\widehat{X}} + \left(\nu Y.\,\widehat{\widehat{Y}} + Z\right)$ |
| sTC$_{\mathbb{N}^{\mathbb{N}}}$ | $\left(\nu X.\widehat{X + \boldsymbol{\Gamma}}\right) \times \left(\zeta X.\widehat{X+1}\right)$ |
| $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ | $\zeta X.\,\widehat{X+1}$ |
| $\Delta^0_1$-FindWS | $\mu X.\,\widehat{\widehat{X}} + \boldsymbol{\Gamma} + 1$ |
| KL | $\zeta X.\,(\nu Y.\,X + Y) \times (\nu Y.\,X + Y)$ |
| lim | $(\nu X.\,X + \widehat{\boldsymbol{\Gamma}}) \times (\zeta X.\,X + 1)$ |
| WKL | $\zeta X.\,1 + X \times X$ |
| $\mathrm{RT}^1_k$ | $\underbrace{(\zeta X.\nu Y.\,X + Y) \times \ldots}_{k \text{ times}}$ |
| LPO′ | $(\nu X.\,\zeta Y.\,X + Y) \times (\zeta X.\,\nu Y.\,X + Y)$ |
| $\mathsf{C}_{\mathbb{N}}$ | $\zeta X.\,\widehat{X+1}$ |
| LPO | $(\nu X.\,X + \boldsymbol{\Gamma}) \times (\zeta X.\,X + 1)$ |
| $\mathsf{C}_k$ | $\underbrace{(\zeta X.\,X + 1) \times \ldots}_{k \text{ times}}$ |
| $\boldsymbol{\Gamma}$ | $\zeta X.\,X$ |
| 0 | $\mu X.\,X$ |

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# Were bundles the right idea?

Perhaps it seemed artificial to introduce bundles $(X_i \neq \emptyset)_{i \in I}$ as our notion of problem?

Why not ask for the category-theoretic version of a *relation*.

**Definition (Relation)**

An *(internal) relation* from an object $X$ to an object $Y$ in a category $\mathcal{C}$ is an object $R$ and a pair of maps $\mathrm{cod}_R : R \to X$, $\mathrm{dom}_R : R \to Y$, such that $\mathrm{cod}_R$ and $\mathrm{dom}_R$ are jointly monic, i.e., $\mathrm{cod}_R \circ f = \mathrm{cod}_R \circ g$ and $\mathrm{dom}_R \circ f = \mathrm{dom}_R \circ g$ implies $f = g$.

Can we define an equivalent notion of reduction between these?

# The opposite of a fibration

The map sending a container $X \to I$ to its shape $I$ is a fibration. In particular, $\mathrm{shape} = \mathrm{cod}^{\mathrm{op}}$.

Spivak[11] suggests that the way to generalise lenses (reductions) is via taking the opposite (dual) of a fibration.

## Definition (Opposite of a Fibration [12])

Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration. Then there is a category $\mathbb{E}^{(\mathrm{op})}$ whose objects are the same as $\mathbb{E}$ but where the arrows $X \to Y$ are (equivalence classes of) spans $X \xleftarrow{f} Z \xrightarrow{g} Y$ such that $f$ is $p$-vertical and $g$ is $p$-cartesian.
There is a fibration $p^{\mathrm{op}} : \mathbb{E}^{\mathrm{op}} \to \mathbb{B}$ sending $X \mapsto p(X)$ and $X \xleftarrow{f} Z \xrightarrow{g} Y \mapsto p(g)$

Property: there is a natural isomorphism of fibres $(\mathbb{E}^{(\mathrm{op})})_I \cong (\mathbb{E}_I)^{(\mathrm{op})}$.

[11] Spivak (2022) — Generalized Lens Categories via functors $\mathcal{C}^{(\mathrm{op})} \to \mathrm{Cat}$
[12] e.g. Streicher (1999) — Fibred Categories à la Jean Bénabou (Section 5)

# Adaptors

## Proposition (Folklore)

*If $\mathcal{C}$ is a category with a factorisation system, then the functor sending a relation $X \xleftarrow{\mathrm{dom}_R} R \xrightarrow{\mathrm{cod}_R} Y$ to its domain $X$ is a fibration.*

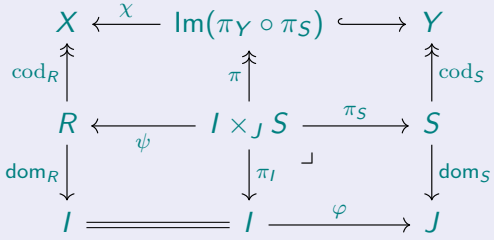
## Definition (Adaptor[13])

The total space of the opposite of the domain functor is the category $\mathsf{DAdt}(\mathcal{C})$ of *adaptors*.

Objects are relations.

Morphisms are (equivalence classes of) diagrams of the form opposite

The top right square arises from the factorisation system.




[13] Braithwaite et al. (2024, unpublished) — Fibre Optics II

# Characterising Strong Reducibility

In $\mathrm{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$, we can factorise an arrow $f = e \circ m$ into a pullback-stable epimorphism (answerable container) followed by a regular monomorphism.

## Definition (Problem)

A *problem* $R : X \to Y$ is a relation $X \xleftarrow{\mathrm{dom}_R} R \xrightarrow{\mathrm{cod}_R} Y$ such that both $\mathrm{dom}_R$ and $\mathrm{cod}_R$ are answerable containers.

## Theorem

*The Strong Weihrauch degrees are isomorphic to the posetal reflection of the category of problems and adaptors over* $\mathrm{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$.

# The algebra of Strong Reducibility

It is possible to define generically:

- Cartesian Products (Meets)
- Tensor Products (Parallel Products)
- Weak Coproducts (Joins)

I have not tried defining composition of strong degrees or anything more complicated!

We can relate containers and adaptors via cylindrification.

# Cylinders

This work is not fully category-theoretic, but relies on $\mathsf{pMod}(\mathbb{A}', \mathbb{A})$.

## Definition (Cylinder)

Given a problem $I \xleftarrow{\pi_I} R \xrightarrow{\pi_X} X$, its *cylinder* is defined to be $\mathsf{Cyl}(R) := \Delta \otimes R$, where $\Delta$ is the adaptor corresponding to the identity function.

## Theorem

*The* $\mathsf{Cyl}$ *functor on adaptors extends to a weak right adjoint to the forgetful functor* $U : \mathit{DAdt}(\mathcal{C}) \to \mathsf{Cont}(\mathcal{C})$.

Caution: The extension of $\mathsf{Cyl}$ to containers isn't functorial!

But we do recover the galois connection between Weihrauch degrees and strong degrees.

# How about joins of strong degrees?

It is possible to give a version of Dzhafarov's[14] construction of joins in the strong degrees that works for $\mathrm{pMod}(\mathbb{A}', \mathbb{A})$.

It relies on "completing" a space by adjoining an infinite element. Categorically, we use a partial map classifier; this exists in $\mathrm{pMod}(\mathbb{A}', \mathbb{A})$.

**Definition ((Regular) Partial Map)**

Let $\mathcal{C}$ be a category with pullbacks. A *partial map* $X \rightharpoonup Y$ is an isomorphism class of spans $X \xleftarrow{f} A \xrightarrow{g} Y$ such that $f$ is a (regular) monomorphism.

**Definition ((Regular) Partial Map Classifier)**

A *(regular) partial map classifier* for an object $Y$ is an object $Y^{\perp}$ and an isomorphism $\mathcal{C}(X, Y^{\perp}) \cong \mathrm{Par}_{\mathcal{C}}(X, Y)$ natural in $X$.

[14] Dzhafarov (2019) — Joins in the strong Weihrauch degrees

# Joins are weak coproducts

**Theorem**

*$DAdt(\mathrm{pMod}(\mathbb{A}', \mathbb{A}))$ has a weak coproduct $\boxplus$.*

**Construction.**

Let $I \xleftarrow{\mathrm{dom}_R} R \xrightarrow{\mathrm{cod}_R} X$ and $J \xleftarrow{\mathrm{dom}_R} S \xrightarrow{\mathrm{cod}_S} Y$ be problems. Define a new problem by:

- The apex is $A = R \times Y^\perp + X^\perp \times S$
- Set $d = [\mathrm{dom}_R \circ \pi_R, \mathrm{dom}_S \circ \pi_S] : R \times Y^\perp + X^\perp \times S \to I + J$
- Set $c = [(\eta_X \circ \mathrm{cod}_R) \times \mathrm{id}_{Y^\perp}, \mathrm{id}_{X^\perp} \times (\eta_Y \circ \mathrm{cod}_S)] : R \times Y^\perp + X^\perp \times S \to X^\perp \times Y^\perp$
- $d$ is a pullback-stable morphism, but $c$ is not, so we factorise it.

So $R \boxplus S = I + J \xleftarrow{d} A \xrightarrow{c'} \mathrm{Im}(c)$. □

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# What we've covered

- Weihrauch reductions are container morphisms
- $\sqcup$, $\sqcap$, $\times$ reflect structure on containers
- Cont(rpReprSp) only a *weak* LCC $\Rightarrow$ weak composition
- Monotone operators on problems $\Rightarrow$ fibred polynomial functors...
- which have fixpoints $\mu$, $\nu$, $\zeta$
- Many natural problems definable using $\zeta$-expressions & Ans
- Strong reductions are adaptors
- We can express cylindrification and weak coproducts

# Questions?




My thesis will be finished Soon™

Also, I am looking for a job!

ian@countingishard.org

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